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Optical mixing in a magnetized plasma

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Abstract. The generation of enhanced electrostatic oscillations in a plasma in a spatially homogeneous magnetic field by two high frequency electromagnetic waves with frequencies ω_1, ω_2 is examined. These enhanced fluctuations may possibly be detected in a light scattering experiment. Starting from a kinetic equation, an expression is found for the stimulated scattering cross section for modes propagating orthogonally to the magnetic field. Resonances at the upper hybrid frequency and harmonics of the cyclotron frequency mean that, if $\omega_1-\omega_2$ is tuned to one of these frequencies, significant enhancement may be expected. The stimulated and thermal scattering cross sections are then compared in the long wavelength limit. It is found that the scattering intensity due to the stimulated cyclotron harmonic modes falls off more rapidly with increasing harmonic number than the scattering intensity in the thermal case. Such a scattering experiment might also be of use as a diagnostic tool in magnetic field measurements in hot laboratory plasmas.

1. Introduction

The scattering of electromagnetic radiation by matter provides a powerful technique for studying the internal structure and dynamics of physical systems and has had wide application in plasma physics since the development of giant-pulse lasers. These experiments are valuable not only in a diagnostic sense (measuring, for example, electron temperature and density) but in providing sensitive tests of plasma theory. The state of laser light scattering by laboratory plasmas has been reviewed by Evans and Katzenstein (1969). The most significant development since this review has been an observation by Evans *et al* (1970) of magnetic modulation in the spectrum of light scattered by a plasma in a strong magnetic field. This work extends the diagnostic value of light scattering experiments since detection of the fine structure induced by the field on the spectrum provides a means of measuring magnetic fields in plasmas.

When collective effects in plasmas are important the spectrum of scattered light contains resonances which correspond to the basic modes of oscillation. For example, in an isotropic plasma a feature appears due to light scattered by eleetron plasma oscillations with frequency ω_p. Although this has been observed, methods have been sought for enhancing the scattered light by generating a suprathermal level of electron plasma waves without at the same time driving the plasma unstable. In one approach two transverse waves with frequencies ω_1, ω_2 are switched on and the frequencies tuned so that $\omega_1-\omega_2 = \omega_3 \simeq \omega_p$. This stimulation procedure is familiar from other branches of physics; in the language of nonlinear optics, Stokes and anti-Stokes scattering modes are associated with each of the pump waves ω_1 and ω_2. A Stokes mode driven for one of these having a frequency $\omega_1-\omega_p$, will on resonance enhance the anti-Stokes scattering of the second pump wave. In the plasma case the geometry must be such that the wavevectors of the transverse waves $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ satisfy $|\boldsymbol{k}_1-\boldsymbol{k}_2|\lambda_D < 1$, where λ_D is the plasma Debye length, to avoid Landau damping of the longitudinal plasma waves.

Experiments have been designed to detect scattering from enhanced plasma oscillations. To date these have been successful at microwave frequencies; Stern and Tzoar (1965) have observed enhanced scattering by using the Tonks–Dattner resonances to excite a suprathermal level of oscillations†. In a magnetized plasma a stimulated scattering experiment may be valuable from a diagnostic point of view. The calculation presented here examines the use of electromagnetic waves to enhance electrostatic oscillations in a plasma in a spatially homogeneous magnetic field. The perturbation scheme is outlined in § 2 and the second order perturbation in plasma density due to nonlinear interactions between the waves is calculated in § 3. Knowing the Fourier transform of the electron density allows us to compute the spectral density of electron density fluctuations $S(\boldsymbol{k}, \omega)$ which in turn determines the scattering cross section. The results are discussed in terms of a scattering experiment in § 4.

2. First order calculation

Consider a two component plasma consisting of electrons and one species of ion; the ions will be treated as stationary, providing a neutralizing uniform background of positive charge. The electron dynamics will be described by the kinetic equation

$$\frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial t}+\boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial \boldsymbol{r}}-\frac{e}{m}\left(\boldsymbol{E}(\boldsymbol{r}, t)+\frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t)\right) \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial \boldsymbol{v}}$$
$$= -\nu(f(\boldsymbol{r}, \boldsymbol{v}, t)-f_0(\boldsymbol{v})) \tag{1}$$

in which collisions have been represented phenomenologically by the right hand side, $f_0(\boldsymbol{v})$ being an equilibrium distribution function and ν a collision frequency. The electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ consists of a switched-on field

$$\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{r}, t) = \boldsymbol{E}_1 \cos(\omega_1 t-\boldsymbol{k}_1 \cdot \boldsymbol{r})+\boldsymbol{E}_2 \cos(\omega_2 t-\boldsymbol{k}_2 \cdot \boldsymbol{r}+\chi)$$

corresponding to two incident waves of frequencies ω_1, ω_2 and wavevectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ with a phase difference χ, together with the selfconsistent electric field in the plasma; $\boldsymbol{B}(\boldsymbol{r}, t)$ represents the magnetic field. In addition we have Poisson's equation

$$\nabla \cdot (\boldsymbol{E}(\boldsymbol{r}, t)-\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{r}, t)) = 4\pi e n_0\left(1-\int f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}\right) \tag{2}$$

in which n_0 represents the equilibrium particle density in the plasma.

If one assumes that the applied electric field does not seriously disturb the plasma one can use the perturbation scheme

$$f = f_0(v)+\epsilon f^{(1)}(\boldsymbol{r}, \boldsymbol{v}, t)+\epsilon^2 f^{(2)}(\boldsymbol{r}, \boldsymbol{v}, t)+ \dots$$
$$\boldsymbol{E} = \epsilon \boldsymbol{E}^{(1)}(\boldsymbol{r}, t)+\epsilon^2 \boldsymbol{E}^{(2)}(\boldsymbol{r}, t)+ \dots$$
$$\boldsymbol{B} = \boldsymbol{B}_0+\epsilon \boldsymbol{B}^{(1)}(\boldsymbol{r}, t)+\epsilon^2 \boldsymbol{B}^{(2)}(\boldsymbol{r}, t)+ \dots$$

where ϵ denotes a perturbation expansion parameter. Here the first order electric field $\boldsymbol{E}^{(1)}(\boldsymbol{r}, t)$ consists of the applied field $\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{r}, t)$ together with the first order selfconsistent field of the plasma.

† A light scattering experiment has now been successful (cf Nodwell R A, Stansfield B L and Meyer J 1971 *Phys. Rev. Lett.* **26** 1219–21).

To zero order, the kinetic equation then becomes (Stix 1962)

$$(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = 0$$

which, on transforming to cylindrical polar coordinates

$$\boldsymbol{v} = (v_\perp \cos\phi, v_\perp \sin\phi, v_\parallel) \tag{3}$$

where $v_\perp$, $v_\parallel$ denote velocity components perpendicular and parallel to $\boldsymbol{B}_0$, implies

$$f_0(\boldsymbol{v}) = f_0(v_\perp, v_\parallel). \tag{4}$$

To first order one finds

$$\frac{\partial f^{(1)}}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f^{(1)}}{\partial \boldsymbol{r}} - \frac{e}{mc}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial f^{(1)}}{\partial \boldsymbol{v}} = -\nu f^{(1)} + \frac{e}{m}\left(\boldsymbol{E}^{(1)} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}^{(1)}\right)\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} \tag{5}$$

with the corresponding Poisson equation

$$\nabla\cdot(\boldsymbol{E}^{(1)} - \boldsymbol{E}^{\mathrm{ext}}) = -4\pi e n_0 \int f^{(1)}\,\mathrm{d}\boldsymbol{v}. \tag{6}$$

Equation (5) may be solved for the first order perturbation to the electron distribution function by integrating along the unperturbed electron orbits. Observe that the left hand side of (5) denotes the rate of change of $f^{(1)}$ as seen by an observer moving on the electron trajectory defined by $\boldsymbol{r} = \boldsymbol{r}(t)$, that is

$$\frac{\mathrm{d}f^{(1)}}{dt} = \frac{\partial f^{(1)}}{\partial t} + \frac{\partial f^{(1)}}{\partial \boldsymbol{r}}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\partial f^{(1)}}{\partial \boldsymbol{v}}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}.$$

The motion of an electron in a uniform magnetic field $\boldsymbol{B}_0$ to zero order is governed by the equations

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v} \qquad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{e}{mc}\boldsymbol{v}\times\boldsymbol{B}_0.$$

Thus the rate of change of $f^{(1)}(\boldsymbol{r}, \boldsymbol{v}, t)$ along the unperturbed electron trajectory is

$$\left(\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t}\right)_{\mathrm{orbit}} = \frac{\partial f^{(1)}}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f^{(1)}}{\partial \boldsymbol{r}} - \frac{e}{mc}\boldsymbol{v}\times\boldsymbol{B}_0\cdot\frac{\partial f^{(1)}}{\partial \boldsymbol{v}}$$

so that (5) becomes

$$\left(\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t}\right)_{\mathrm{orbit}} = -\nu f^{(1)} + \frac{e}{m}\left(\boldsymbol{E}^{(1)} + \frac{\boldsymbol{v}\times\boldsymbol{B}^{(1)}}{c}\right)\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{7}$$

Integrating along the zero order electron trajectory from $t' = -\infty$ up to $t' = t$ gives

$$f^{(1)}(\boldsymbol{r}, \boldsymbol{v}, t) = \frac{e}{m}\int_{-\infty}^{t}\left(\boldsymbol{E}^{(1)}(\boldsymbol{r}', t') + \frac{\boldsymbol{v}'\times\boldsymbol{B}^{(1)}(\boldsymbol{r}', t')}{c}\right)\cdot\frac{\partial f_0(\boldsymbol{v}')}{\partial \boldsymbol{v}'}\exp\{-\nu(t-t')\}\,\mathrm{d}t'. \tag{8}$$

The electron trajectory $\boldsymbol{r}'(t')$ ending at $\boldsymbol{r}' = \boldsymbol{r}$ at $t' = t$ is given by

$$x' = \frac{v_\perp}{\Omega}[\sin\{\Omega(t'-t)+\phi\} - \sin\phi] + x$$

$$y' = -\frac{v_\perp}{\Omega}[\cos\{\Omega(t'-t)+\phi\} - \cos\phi] + y$$

$$z' = v_\parallel(t'-t) + z \tag{9}$$

with $\Omega = eB_0/mc$, the electron cyclotron frequency. From $\mathrm{d}\boldsymbol{r}'/\mathrm{d}t' = \boldsymbol{v}'$ one has

$$v'_x = v_\perp \cos\{\Omega(t'-t)+\phi\}$$
$$v'_y = v_\perp \sin\{\Omega(t'-t)+\phi\}$$
$$v'_z = v_\parallel . \tag{10}$$

From (10) it is clear that $(v'_\perp)^2 \equiv (v'_x)^2+(v'_y)^2 = v_\perp^2$ so that $v_\perp^2$ and $v_\parallel$ are constants of the motion. Hence, from (4), $f_0(v_\perp, v_\parallel)$, $\partial f_0/\partial v_\perp$ and $\partial f_0/\partial v_\parallel$ have the same form in primed and unprimed coordinates. Comparing (3) and (10) shows that

$$\phi' = \Omega(t'-t)+\phi$$

and hence

$$\frac{\partial}{\partial \boldsymbol{v}'} \equiv (\boldsymbol{i}\cos(\phi-\Omega\tau)+\boldsymbol{j}\sin(\phi-\Omega\tau))\frac{\partial}{\partial v_\perp}+\hat{\boldsymbol{z}}\frac{\partial}{\partial v_\parallel}$$
$$+\frac{1}{v_\perp}(-\hat{\boldsymbol{i}}\sin(\phi-\Omega\tau)+\hat{\boldsymbol{j}}\cos(\phi-\Omega\tau))\frac{\partial}{\partial \phi'}$$

where $\tau = t-t'$.

Fourier transforming equations (8) and (6) gives

$$f^{(1)}(\boldsymbol{k}, \boldsymbol{v}, \omega) = \frac{e}{m}\int_0^\infty \mathrm{d}\tau \exp\{\mathrm{i}(\omega+\mathrm{i}\nu)\tau+\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{r}'(\tau)-\boldsymbol{r})\}\boldsymbol{E}^{(1)}(\boldsymbol{k}, \omega)\,.\,\frac{\partial f_0(\boldsymbol{v}'(\tau))}{\partial \boldsymbol{v}'(\tau)} \tag{11}$$

$$\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{k}, \omega)-\boldsymbol{E}^{(1)}(\boldsymbol{k}, \omega)) = 4\pi e n_0 \int f^{(1)}(\boldsymbol{k}, \boldsymbol{v}, \omega)\,\mathrm{d}\boldsymbol{v}. \tag{12}$$

Substituting $f^{(1)}$ from (11) in (12), Poisson's equation gives

$$\boldsymbol{k}\,.\,\boldsymbol{E}^{(1)}(\boldsymbol{k}, \omega) = \frac{\boldsymbol{k}\,.\,\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{k}, \omega)}{\epsilon(\boldsymbol{k}, \omega)}$$

where ν has been chosen to be velocity independent, and $\epsilon(\boldsymbol{k}, \omega)$ is the dielectric function of the plasma given by

$$\epsilon(\boldsymbol{k}, \omega) = 1-\frac{\mathrm{i}\omega_p^2}{k^2}\int\int \mathrm{d}\tau\,\mathrm{d}\boldsymbol{v} \exp\{\mathrm{i}(\omega+\mathrm{i}\nu)\tau+\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{r}'(\tau)-\boldsymbol{r})\}\boldsymbol{k}\,.\,\frac{\partial f_0(\boldsymbol{v}')}{\partial \boldsymbol{v}'}.$$

Writing $\boldsymbol{k} = (k_\perp \cos\alpha, k_\perp \sin\alpha, 0)$ and noting that

$$\exp\{\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{r}'(\tau)-\boldsymbol{r})\} = \exp(\mathrm{i}a\sin(\phi-\alpha-\Omega\tau)-\mathrm{i}a\sin(\phi-\alpha))$$
$$= \exp(-\mathrm{i}a\sin(\phi-\alpha))\sum_{n=-\infty}^{\infty} \mathrm{J}_n(a)\exp\{\mathrm{i}n(\phi-\alpha-\Omega\tau)\}$$

where J_n is a Bessel function of the first kind or order n, and $a = k_\perp v_\perp/\Omega$, one has from (12)

$$f^{(1)}(\boldsymbol{k}, \boldsymbol{v}, \omega) = \frac{e}{2m}\frac{1}{\epsilon(k_\perp, \omega)}\sum\sum_{n,p=-\infty}^{\infty} \exp\{-\mathrm{i}(n-p)\alpha\}$$
$$\times\left\{\frac{\partial f_0}{\partial v_\perp}\left((\mathrm{i}E_x^{\mathrm{ext}}+E_y^{\mathrm{ext}})\frac{\exp\{\mathrm{i}(n-p+1)\phi\}}{\omega-\overline{n+1}\Omega}\right.\right.$$

$$+ (\mathrm{i}E_x^{\mathrm{ext}} - E_y^{\mathrm{ext}}) \frac{\exp\{\mathrm{i}(n-p-1)\phi\}}{\omega - \overline{n-1}\Omega}\Bigg)$$

$$+ 2\frac{\partial f_0}{\partial v_{\parallel}}\mathrm{i}E_z^{\mathrm{ext}}\frac{\exp\{\mathrm{i}(n-p)\phi\}}{\omega - n\Omega}\Bigg\}\mathrm{J}_n(a)\mathrm{J}_p(a) \tag{13}$$

where ω has been written for $\omega + \mathrm{i}\nu$ in the denominators and dielectric function in (13). The dielectric function $\epsilon(k_\perp, \omega)$ may be represented in the form

$$\epsilon(k_\perp, \omega) = 1 - \frac{1}{(k_\perp \lambda_\mathrm{D})^2} \sum_{n=-\infty}^{\infty} \frac{n\Omega}{\omega - n\Omega} \mathrm{e}^{-\lambda}\mathrm{I}_n(\lambda) \tag{14}$$

where $\lambda_\mathrm{D} = (kT/4\pi n_0 e^2)^{1/2}$ is the Debye length in a plasma of density n_0 and temperature T and $\mathrm{I}_n(\lambda)$ is the modified Bessel function of the first kind, with argument $= k_\perp^2 kT/\Omega^2 m$. The transform of the switched-on field is given by

$$\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{k}, \omega) = \tfrac{1}{2}\boldsymbol{E}_1\{\delta(\omega + \omega_1)\delta(\boldsymbol{k} + \boldsymbol{k}_1)$$

$$+ \tfrac{1}{2}\boldsymbol{E}_2\{\delta(\omega + \omega_2)\delta(\boldsymbol{k} + \boldsymbol{k}_2)\,\mathrm{e}^{\mathrm{i}\chi} + \delta(\omega - \omega_2)(\boldsymbol{k} - \boldsymbol{k}_2)\,\mathrm{e}^{-\mathrm{i}\chi}\} \tag{15}$$

and it has been assumed that the electron temperatures across and along the magnetic field are the same, so that

$$f_0 = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp - \left(\frac{m(v_\perp^2 + v_\parallel^2)}{2kT}\right).$$

3. Second order calculation

Extending the perturbation calculation to second order gives

$$\left(\frac{\mathrm{d}f^{(2)}}{\mathrm{d}t}\right)_{\mathrm{orbit}} = -\nu f^{(2)} + \frac{e}{m}\left\{\left(\boldsymbol{E}^{(1)}(\boldsymbol{r}, t) + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}^{(1)}(\boldsymbol{r}, t)\right) . \frac{\partial f^{(1)}}{\partial \boldsymbol{v}} + \boldsymbol{E}^{(2)}(\boldsymbol{r}, t) . \frac{\partial f_0}{\partial \boldsymbol{v}}\right\}$$

$$-\nabla . \boldsymbol{E}^{(2)}(\boldsymbol{r}, t) = \nabla^2 \phi^{(2)}(\boldsymbol{r}, t) = 4\pi e n_0 \int f^{(2)}(\boldsymbol{r}, \boldsymbol{v}, t)\,\mathrm{d}\boldsymbol{v} \tag{17}$$

and so the second order number density

$$n^{(2)}(\boldsymbol{r}, t) = \frac{\nabla^2 \phi^{(2)}(\boldsymbol{r}, t)}{4\pi e}. \tag{18}$$

By integrating along unperturbed orbits as in the first order calculation, one finds on Fourier transforming

$$f^{(2)}(\boldsymbol{k}, \boldsymbol{v}, \omega) = \frac{e}{m}\exp\{-\mathrm{i}(\boldsymbol{k} . \boldsymbol{r} - \omega t)\} \int\int \mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}' \int_{-\infty}^{t} \mathrm{d}t' \exp\{\mathrm{i}(\boldsymbol{k} . \boldsymbol{r}' - \omega t')\}$$

$$\times \left(\boldsymbol{E}^{(1)}(\boldsymbol{k} - \boldsymbol{k}', \omega - \omega') + \frac{\boldsymbol{v}'}{\omega - \omega'} \times \{(\boldsymbol{k} - \boldsymbol{k}') \times \boldsymbol{E}^{(1)}(\boldsymbol{k} - \boldsymbol{k}', \omega - \omega')\}\right)$$

$$. \frac{\partial f^{(1)}(\boldsymbol{k}', \boldsymbol{v}', \omega')}{\partial \boldsymbol{v}'}$$

$$+ \frac{\mathrm{i}e}{m}\phi^{(2)}(\boldsymbol{k}, \omega) \int_{-\infty}^{t} \mathrm{d}t' \exp\{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}' - \boldsymbol{r}) + \mathrm{i}\omega(t - t')\}\boldsymbol{k} . \frac{\partial f_0(\boldsymbol{v}')}{\partial \boldsymbol{v}'}. \tag{19}$$

Denoting the first term on the right hand side of (19) by $(e/m)H$, the Fourier transform of Poisson's equation gives on substituting for $f^{(2)}$ from (19)

$$k^2\phi^{(2)}(\boldsymbol{k},\omega) = \omega_p^2 \int H\,d\boldsymbol{v} + i\omega_p^2 k\phi^{(2)}(\boldsymbol{k},\omega)\int_0^\infty v_\perp \frac{df_0}{dv_\perp}dv_\perp \int_0^{2\pi} d\phi \int_0^\infty d\tau \cos(\phi-\alpha-\Omega\tau)\,e^{i\rho}$$

where

$$\rho = \omega\tau + a\sin(\phi-\alpha-\Omega\tau) - a\sin(\phi-\alpha)$$

so that

$$\phi^{(2)}(\boldsymbol{k},\omega) = \frac{\omega_p^2 \int H\,d\boldsymbol{v}}{k_\perp^2 \epsilon(k_\perp,\omega)} \tag{20}$$

with $\epsilon(k_\perp.\,\omega)$ given by (13).

Thus, using (18)

$$n^{(2)}(\boldsymbol{k},\omega) = \frac{n_0 e}{m}\frac{\int H\,d\boldsymbol{v}}{\epsilon(k_\perp,\omega)}. \tag{21}$$

The details of the calculation are shown in the Appendix.

The expression for $n^{(2)}(\boldsymbol{k},\omega)$ is then

$$n^{(2)}(\boldsymbol{k},\omega) = \frac{n_0 e^2 \pi}{2m^2}\int v_\perp\,dv_\perp \int\int \frac{d\omega'\,d\boldsymbol{k}'}{\epsilon(k_\perp,\omega)}\sum_{s=-\infty}^{\infty}\frac{\exp\{is(\alpha+\beta+\theta)\}}{\omega'(\omega-s\Omega)}J_s(a)$$
$$\times\Bigg[\Bigg\{\left(\frac{mv_\perp}{kT}\right)^2 f_0[E_\perp(\boldsymbol{k}',\omega')E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\exp\{-2i(\beta+\theta)\}J_{s-2}(b)$$
$$+E_\perp^*(\boldsymbol{k}',\omega')E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\exp\{2i(\beta+\theta)\}J_{s+2}(b)]$$
$$-2\left(\frac{d^2 f_0}{dv_\perp^2}+\frac{1}{v_\perp}\frac{df_0}{dv_\perp}\right)J_s(b)\,\mathrm{Re}(E_\perp(\boldsymbol{k}',\omega')E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))\Bigg\}$$
$$+\frac{2|\boldsymbol{k}-\boldsymbol{k}'|}{\omega-\omega'}\frac{df_0}{dv_\perp}\Bigg(E_z(\boldsymbol{k}',\omega')E_z(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')$$
$$\times[\exp\{-i(\bar{\alpha}+\beta+\theta)\}J_{s-1}(b)+\exp\{i(\bar{\alpha}+\beta+\theta)\}J_{s+1}(b)]$$
$$+2if_0(v_\perp)\frac{k'_\perp}{\Omega}E_z(\boldsymbol{k}',\omega')E_z(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')J_s(b)\sin(\bar{\alpha}-\alpha')$$
$$+\frac{df_0}{dv_\perp}[E_\perp(\boldsymbol{k}',\omega')\exp\{-i(\beta+\theta)\}J_{s-1}(b)$$
$$+E_\perp^*(\boldsymbol{k}',\omega')\exp\{i(\beta+\theta)\}J_{s+1}(b)]$$
$$\times\mathrm{Re}(e^{i\bar{\alpha}}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))\Bigg)\Bigg]. \tag{22}$$

On setting $E_z = 0$ and $\alpha = \beta = \theta = 0$ so that $E_x = 0$, the one dimensional result for the second order perturbation becomes

$$n^{(2)}(\boldsymbol{k},\omega) = \frac{n_0 e^2 E_1 E_2}{4m^2 v_e^2 \omega_1\omega_2\epsilon(k,\omega)}\sum_{s=-\infty}^{\infty}\frac{2\Omega Q(s)-\omega R(s)}{\omega-s\Omega}\sum_{\pm}$$
$$\times\,\delta(\omega\pm\Delta\omega)\delta(\boldsymbol{k}\pm\Delta\boldsymbol{k}) \tag{23}$$

where

$$Q(s) \equiv \int_0^\infty dx\, x \exp(-x^2) s J_s^2(\mu x) = \frac{s}{2} e^{-\lambda} I_s(\lambda)$$

$$R(s) \equiv \int_0^\infty dx \exp(-x^2) J_s(\mu x)(4x^3 J_s''(\mu x) + 2x J_s(\mu x))$$

$$= e^{-\lambda} I_s(\lambda)\left(\frac{s^2}{\lambda} + 2\lambda\right) - (1 + 2\lambda) e^{-\lambda} I_s'(\lambda)$$

$$\mu = \frac{k_\perp v_e \sqrt{2}}{\Omega} \qquad v_e = \left(\frac{kT}{m}\right)^{1/2} \qquad x = \frac{v_\perp}{v_e \sqrt{2}}$$

and $\Delta\omega = \omega_2 - \omega_1$, $\Delta \boldsymbol{k} = \boldsymbol{k}_2 - \boldsymbol{k}_1$. This result agrees with that of Sauer and Wallis (1966). In the limit $T \to 0$

$$n_{T=0}^{(2)}(\boldsymbol{k}, \omega) = \frac{3 n_0 e^2 E_1 E_2 k^2 \Sigma_\pm \delta(\omega \pm \Delta\omega)\delta(\boldsymbol{k} \pm \Delta\boldsymbol{k})}{m^2 \omega_1 \omega_2 \epsilon_0(k, \omega)(\omega^2 - \Omega^2)}$$

where $\epsilon_0(k, \omega) = 1 - \omega_p^2/(\omega^2 - \Omega^2)$; this agrees with the cold plasma calculation of Weyl (1970) apart from the factor 3.

The electrostatic modes driven by the applied fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ may be observable in a light scattering experiment, as suggested by Kroll *et al* (1964) in the isotropic case. One can then define a differential scattering cross section per unit frequency interval per unit solid angle per electron

$$\frac{d^2\sigma}{d\omega\, d\Sigma} = \frac{1}{4\pi} S(\boldsymbol{k}, \omega)(1 + \cos^2 \Theta) r_e^2 \tag{24}$$

where Θ is the scattering angle and $r_e = e^2/mc^2$ is the classical electron radius. $S(\boldsymbol{k}, \omega)$ is the scattering form factor defined by

$$S(\boldsymbol{k}, \omega) = \lim_{V, T \to \infty} \frac{\langle 2|n^{(2)}(\boldsymbol{k}, \omega|^2\rangle}{n_0 V T} \tag{25}$$

(Evans and Katzenstein 1969), where V is the plasma volume, n_0 the electron density and the brackets denote an ensemble average. Using (23) one finds for the form factor

$$S(\boldsymbol{k}, \omega) = \frac{n_0 e^4 (E_1 E_2)^2}{8 m^4 v_e^4 \omega_1^2 \omega_2^2} \left(\sum_{s=-\infty}^{\infty} \frac{2\Omega Q(s)(\omega - s\Omega) - \{(\omega - s\Omega)\omega + \nu^2\} R(s)}{(\omega - s\Omega)^2 + \nu^2} \right)^2 \times \frac{\Sigma_\pm \delta(\omega \pm \Delta\omega)\delta(\boldsymbol{k} \pm \Delta\boldsymbol{k})}{|\epsilon(\boldsymbol{k}, \tilde{\omega})|^2} \tag{26}$$

where ω is now real, $\tilde{\omega} = \omega + i\nu$ and $|\omega - s\Omega| > \nu$. The cross section integrated over frequency may then be compared with the thermal cross sections per unit solid angle for scattering at both the upper-hybrid resonance $\omega_{UH} = (\omega_p^2 + \Omega^2)^{1/2}$ and the nth Bernstein mode; these are

$$\left[\frac{d\sigma(\omega_{UH})}{d\Sigma}\right]_{th} = \frac{1}{(2\pi)^4} \frac{\lambda (k\lambda_D)^2}{\lambda + (k\lambda_D)^2} r_e^2 \tag{27a}$$

$$\left[\frac{d\sigma(n\Omega)}{d\Sigma}\right]_{th} = \frac{1}{(2\pi)^4} \frac{\lambda^n r_e^2}{2^n n!(1 - \lambda)} \tag{27b}$$

where $\lambda \ll 1$ (Salpeter 1961) and the angular factor has been omitted.

4. Discussion

The warm plasma theory developed in § 3 provides a scattering form factor $S(\boldsymbol{k}, \omega)$ which depends on the applied electric fields, the plasma dielectric function $\epsilon(\boldsymbol{k}, \omega)$ and a complicated amplitude expressed as a sum over a harmonic number s. The one dimensional result (23) will serve to illustrate the main features of the general expression.

Kroll *et al* (1964) showed that oscillations in unmagnetized plasmas driven by coupled radiation fields can lead to scattering cross sections greatly enhanced over thermal levels. Subsequently Bloembergen and others pointed out that the enhancement factors ($\simeq 10^{13}$) were more apparent than real on account of the fact that the cross section per unit solid angle was estimated whereas for diffraction limited beams the available solid angle is very small. Nonetheless the facts that a real enhancement has been demonstrated by Stern and Tzoar at microwave frequencies and that Evans and Carolan (1970) have reported light scattering by magnetized thermal plasmas make it timely to consider a stimulated laser scattering experiment for a plasma in a magnetic field, not least in view of its promise as far as magnetic field diagnostics are concerned.

First one may compute the enhancement factor for a typical laboratory plasma. The cross section for stimulated scattering is obtained by integrating (24) over frequency, that is, replacing ω by $\Delta\omega$. When $\Delta\omega \simeq \omega_{\text{resonant}}$ which satisfies the dispersion relation, significant enhancement could be expected. For a warm plasma in which electrostatic waves propagate at right angles to the magnetic field, the roots of the dispersion relation occur at the upper hybrid frequency and close to the cyclotron harmonics for $\lambda \ll 1$. At the upper hybrid resonance

$$|\epsilon(k, \tilde{\omega})|^2 \to \Gamma^2 \qquad \Gamma = \frac{2\nu\omega_{\text{UH}}}{\omega_{\text{p}}^2}$$

and

$$\frac{[\mathrm{d}\sigma(\omega_{\text{UH}})/\mathrm{d}\Sigma]_{\text{stim}}}{[\mathrm{d}\sigma(\omega_{\text{UH}})/\mathrm{d}\Sigma]_{\text{th}}} = 72n_0\left(\frac{e^2E_1E_2k^2}{m^2\Gamma\omega_1\omega_2\omega_{\text{p}}^2}\right)^2\frac{\lambda+k^2\lambda_{\text{D}}^2}{\lambda k^2\lambda_{\text{D}}^2}\sum_{\pm}\delta(\boldsymbol{k}\pm\Delta\boldsymbol{k}).$$

At the cyclotron harmonic $\omega \simeq r\Omega$ keeping only the rth term in the dispersion relation gives

$$\omega - r\Omega \simeq \left(\frac{\omega_{\text{p}}}{\Omega}\right)^2\frac{\lambda^{r-1}\Omega}{(r-1)!2^r}.$$

Note that for relativistic corrections to be negligible we require $\omega - r\Omega \equiv \Omega\delta_r \gg r\Omega v_{\text{e}}^2/c^2$. Then

$$\frac{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{stim}}}{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{th}}} \simeq \frac{n_0e^4(E_1E_2)^2 2^r r!}{m^4v_{\text{e}}^4\omega_1^2\omega_2^2\Gamma^2\lambda^r}$$
$$\times\left(\frac{2Q(r)-rR(r)}{\delta_r}+\sum_{\substack{s=-\infty\\ s\neq r}}^{\infty}\frac{2Q(s)-rR(s)}{r-s}\right)^2\sum_{\pm}\delta(\boldsymbol{k}\pm\Delta\boldsymbol{k}). \tag{28}$$

Noting that $Q(s) \simeq \lambda^s/2^{s+1}(s-1)!$, $s \geqslant 1$ and $R(s) \simeq \lambda^{s-1}/2^s(s-2)!$, $s \geqslant 2$ for $\lambda \ll 1$ the dominant contribution comes from the first term in the bracket so that

$$\frac{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{stim}}}{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{th}}} \simeq \frac{r^3(r-1)n_0}{2^{r+1}(r-2)!}\left(\frac{e^2E_1E_2\Omega}{m^2v_{\text{e}}^2\omega_1\omega_2\nu}\right)^2\lambda^{r-2}\sum_{\pm}\delta(\boldsymbol{k}\pm\Delta\boldsymbol{k})$$

since $\Gamma \simeq (\nu\Omega/\omega_p^2)2^r(r-1)!\,\lambda^{1-r}$. In practice the incident beams have a finite spread in wavenumber so that the delta function must be replaced by an interaction volume V which may be taken as the volume of intersection of the two beams. Choosing $n_0 = 10^{14}\,\text{cm}^{-3}$, $T = 50\,\text{eV}$, $B_0 = 7{\cdot}3\,\text{kG}$, $\nu/\Omega = 10^{-2}$ and using ruby lasers with wavelength 7×10^{-5} cm and fields of $10^8\,\text{V cm}^{-1}$ one has $\sigma_{\text{stim}}(\omega_{\text{UH}})/\sigma_{\text{th}}(\omega_{\text{UH}}) \simeq 10^{12}$ for an interaction volume $10^{-3}\,\text{cm}^3$. In practice of course this factor will be greatly reduced as Bloembergen and Shen have pointed out in commenting on the KRR result.

The details of the spectrum for stimulated scattering are of greater interest, in particular the relative intensities of the radiation scattered into consecutive harmonics. From (28) it follows that

$$\frac{[\mathrm{d}\sigma(\overline{r+1}\Omega)/\mathrm{d}\Sigma]_{\text{stim}}}{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{stim}}} \simeq \frac{1}{4}\left(\frac{r+1}{r-1}\right)^2\frac{\lambda^2}{r^2}$$

whereas for the thermal spectrum

$$\frac{[\mathrm{d}\sigma(\overline{r+1}\Omega)/\mathrm{d}\Sigma]_{\text{th}}}{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{th}}} = \frac{\lambda}{2(r+1)}.$$

Thus, as the harmonic number r increases the scattering due to the driven Bernstein mode decays as $(\lambda/r)^2$, that is, more rapidly than in the thermal spectrum. A similar comparison may be made between the upper hybrid resonance and the rth Bernstein mode for the two spectra. Platzman *et al* (1968) considered some details of the spectrum of scattered light from a magnetized plasma in the thermal case in the small λ regime. For $k\lambda_D < 1$ and $\boldsymbol{k} \perp \boldsymbol{B}_0$ the thermal spectrum consists of a series of peaks corresponding to the upper hybrid resonance and the Bernstein modes located near the cyclotron harmonics $r\Omega (r \geqslant 2)$. The relative intensities of light scattered by the rth order Bernstein mode to that scattered by the upper hybrid is λ^r. Comparing these features in the stimulated spectrum gives

$$\frac{[\mathrm{d}\sigma(r\Omega)/\mathrm{d}\Sigma]_{\text{stim}}}{[\mathrm{d}\sigma(\omega_{\text{UH}})/\mathrm{d}\Sigma]_{\text{stim}}} \simeq \frac{r^2}{2^{2r+4}\{(r-2)!\}^2}\frac{\omega_{\text{UH}}^2}{\Omega^2}\lambda^{2r-4} \qquad r \geqslant 2$$

which is independent of λ for $r = 2$. For the plasma considered earlier the condition $\omega - r\Omega \gg r\Omega v_e^2/c^2$ implies that $r \not> 2$ in any case for a nonrelativistic treatment to be valid. In this case

$$\frac{[\mathrm{d}\sigma(2\Omega)/\mathrm{d}\Sigma]_{\text{stim}}}{[\mathrm{d}\sigma(\omega_{\text{UH}})/\mathrm{d}\Sigma]_{\text{stim}}} \simeq \tfrac{1}{2}$$

in sharp contrast with the behaviour of the thermal spectrum. In that case Platzman *et al* concluded that in the long wavelength limit it will be difficult to observe scattering from the Bernstein modes and suggested choosing the magnetic field so that one of these modes becomes degenerate with the upper hybrid to allow a transfer of intensity into the degenerate mode. This would not be straightforward in practice since it requires arranging the magnetic field so that $(r^2-1)\Omega^2 = \omega_p^2$, $r \geqslant 2$ that is, measuring the plasma density beforehand.

The attraction of a scattering experiment using Bernstein modes enhanced by applied electric fields over its counterpart in an isotropic plasma lies principally in the fact that magnetic field uniformity is demanded rather than plasma homogeneity. This condition may be more easily realized experimentally. Moreover, tuning the system by varying the magnetic field is certainly feasible while in the isotropic plasma experiment it would hardly be possible since the plasma density would have to be varied to make $\omega_1 - \omega_2 \simeq \omega_p$. The possibility of adapting the mixing experiments for diagnostic purposes is worth examining. This demands of course a laser system that may be readily tuned to a cyclotron harmonic resonance to provide a measurement of magnetic fields in hot dense plasmas. In this connection it is worth noting that Nodwell *et al* (1970) have constructed such a laser system operating, however, at low powers.

We conclude that warm plasma theory predicts an enhancement in the intensity of light scattered by driven Bernstein modes in a magnetized plasma over the thermal spectrum. Moreover, there are differences in the scattered light spectra between stimulated and thermal oscillations. In particular the intensity scattered by the mode at 2Ω is comparable with that due to the upper hybrid frequency in contrast with the thermal spectrum. This may be significant for a stimulated scattering experiment in that it should be easier to arrange $\omega_2 - \omega_1 = 2\Omega$ than to tune to the upper hybrid frequency which is a function of plasma density as well as of magnetic field.

Finally, the discussion in this paper has been restricted to electrostatic waves with $\boldsymbol{k} \perp \boldsymbol{B}_0$ while in an actual experiment one would collect scattered light from a cone of $\boldsymbol{k}$ vectors about $\boldsymbol{k}_\perp$. However, it is worth noting that Carolan and Evans (1971) have recently shown that the composite scattered light spectrum is strongly influenced by the contribution from $\boldsymbol{k} = \boldsymbol{k}_\perp$.

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Appendix

Observing that

$$\frac{\partial f^{(1)}(\boldsymbol{k}', \boldsymbol{v}', \omega')}{\partial \boldsymbol{v}'} = \hat{\boldsymbol{i}}\left(\cos(\phi - \Omega\tau)\frac{\partial f^{(1)}}{\partial v_\perp} - \frac{1}{v_\perp}\sin(\phi - \Omega\tau)\frac{\partial f^{(1)}}{\partial \phi'}\right) + \hat{\boldsymbol{j}}\left(\sin(\phi - \Omega\tau)\frac{\partial f^{(1)}}{\partial v_\perp} + \frac{1}{v_\perp}\cos(\phi - \Omega\tau)\frac{\partial f^{(1)}}{\partial \phi'}\right) + \hat{\boldsymbol{z}}\frac{\partial f^{(1)}}{\partial v_\parallel}$$

and defining

$$G_1 = \frac{ie}{2m\epsilon(\boldsymbol{k}', \omega')} \sum_{n,p=-\infty}^{\infty}\sum \exp\{-\mathrm{i}(n-p)\alpha'\} \frac{\partial f_0}{\partial v_\perp} \mathrm{J}_n(a')\mathrm{J}_p(a')$$

$$G_2 = -\mathrm{i}\frac{\partial G_1}{\partial v_\perp} \qquad G_3 = 2\frac{\partial}{\partial v_\perp}\left(\frac{v_\parallel}{v_\perp}G_1\right) \qquad G_4 = 2\mathrm{i}\frac{v_\parallel}{v_\perp}G_1$$

$$A_{p,j} = \frac{1}{\omega' - j\Omega}\exp\{\mathrm{i}(j-p)(\phi - \Omega\tau)\}$$

where

$$\boldsymbol{k}' = (k'_\perp \cos\alpha', k'_\perp \sin\alpha', 0) \qquad a = \frac{k'_\perp v_\perp}{\Omega}$$

it follows that

$$\frac{\partial f^{(1)}}{\partial v_\perp} = G_2\{A_{p,n+1}(\mathrm{i}E_x^{\mathrm{ext}}(\boldsymbol{k}',\omega')+E_y^{\mathrm{ext}}(\boldsymbol{k}',\omega'))$$

$$+A_{p,n-1}(\mathrm{i}E_x^{\mathrm{ext}}(\boldsymbol{k}',\omega')-E_y^{\mathrm{ext}}(\boldsymbol{k}',\omega'))\}+G_3A_{p,n}E_z^{\mathrm{ext}}(\boldsymbol{k}',\omega')$$

$$\frac{\partial f^{(1)}}{\partial \phi'} = G_1\{(n-p+1)A_{p,n+1}(\mathrm{i}E_x^{\mathrm{ext}}(\boldsymbol{k}',\omega')+E_y^{\mathrm{ext}}(\boldsymbol{k}',\omega'))+(n-p-1)$$

$$\times(\mathrm{i}E_x^{\mathrm{ext}}(\boldsymbol{k}',\omega')-E_y^{\mathrm{ext}}(\boldsymbol{k}',\omega'))\}+G_4(n-p)A_{p,n}E_z^{\mathrm{ext}}(\boldsymbol{k}',\omega').$$

The term $\partial f^{(1)}/\partial v_\parallel$ is not displayed since its integral over velocity space is zero. Further, defining

$$B = \exp(-\mathrm{i}a\sin(\phi-\alpha))\sum_{r=-\infty}^{\infty}\mathrm{J}_r(a)\exp\{-\mathrm{i}r(\alpha-\phi)\}$$

$$Q_t^j = \frac{\exp\{\mathrm{i}(n-p+t)\phi\}}{(\omega'-j\Omega)(\omega-n-p+r+t\Omega)}$$

and

$$E_\perp^{\mathrm{ext}} = \mathrm{i}E_x^{\mathrm{ext}}+E_y^{\mathrm{ext}}$$

the τ integration may be carried out, giving, after some lengthy algebra

$$H = \int\int \mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}'\frac{B}{2\epsilon(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')}\Bigg([G_2\{E_\perp(\boldsymbol{k}',\omega')$$

$$\times(Q_2^{n+1}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')-Q_0^{n+1}E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))$$

$$-E_\perp^*(\boldsymbol{k}',\omega')(Q_0^{n-1}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')-Q_{-2}^{n-1}E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))\}$$

$$+\frac{\mathrm{i}}{v_\perp}G_1\{(n-p+1)E_\perp(\boldsymbol{k}',\omega')$$

$$\times(Q_2^{n+1}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')+Q_0^{n+1}E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))$$

$$-(n-p-1)E_\perp^*(\boldsymbol{k}',\omega')(Q_0^{n-1}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')$$

$$+Q_{-2}^{n-1}E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))\}]$$

$$+\frac{|\boldsymbol{k}-\boldsymbol{k}'|}{\omega-\omega'}\Bigg[-\mathrm{i}G_1[(\mathrm{e}^{\mathrm{i}\bar\alpha}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')$$

$$+\mathrm{e}^{-\mathrm{i}\bar\alpha}E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))$$

$$\times\{(n-p+1)E_\perp(\boldsymbol{k}',\omega')Q_1^{n+1}-(n-p-1)E_\perp^*(\boldsymbol{k}',\omega')Q_{-1}^{n-1}\}]$$

$$+E_z(\boldsymbol{k}',\omega')E_z(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')$$

$$\times\left\{\mathrm{e}^{-\mathrm{i}\bar\alpha}Q_1^n\left(\mathrm{i}v_\parallel G_3-(n-p)\frac{v_\parallel}{v_\perp}G_4\right)\right.$$

$$\left.+\mathrm{e}^{\mathrm{i}\bar\alpha}Q_{-1}^n\left(\mathrm{i}v_\parallel G_3+(n-p)\frac{v_\parallel}{v_\perp}G_4\right)\right\}\Bigg]\Bigg) \tag{A.1}$$

with $\boldsymbol{k}-\boldsymbol{k}' = |\boldsymbol{k}-\boldsymbol{k}'|(\cos\bar\alpha, \sin\bar\alpha, 0)$. The term $E_\perp^*$ denotes the complex conjugate of $E_\perp$

and we have dropped the superscript label 'ext'. In all subsequent working $E_\perp$ denotes $E_\perp^{\text{ext}}$.

We now substitute this expression for H, to determine $n^{(2)}(\boldsymbol{k},\omega)$ and find, after integrating over $v_\parallel$ and ϕ

$$\begin{aligned}
n^{(2)}(\boldsymbol{k},\omega) &= \frac{n_0 e^2 \pi}{2m^2}\int v_\perp\, dv_\perp \int\int \frac{d\omega'\, d\boldsymbol{k}'}{\epsilon(k'_\perp,\omega')\epsilon(k_\perp-k'_\perp,\omega-\omega')\epsilon(k_\perp,\omega)} \\
&\times \sum\sum\sum_{n,p,s=-\infty}^{\infty} \exp\{-i(n-p)(\alpha'-\alpha)\}\frac{J_s(a)}{\omega - s\Omega} \\
&\times \Bigg(\Bigg\{E_\perp(\boldsymbol{k}',\omega')E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{J_{s-n+p-2}(a)\, e^{2i\alpha}}{\omega'-\overline{n+1}\Omega}\left(\frac{\partial F_{np}}{\partial v_\perp} - \frac{n-p+1}{v_\perp}F_{np}\right) \\
&- E_\perp(\boldsymbol{k}',\omega')E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{J_{s-n+p}(a)}{\omega'-\overline{n+1}\Omega}\left(\frac{\partial F_{np}}{\partial v_\perp} + \frac{n-p+1}{v_\perp}F_{np}\right) \\
&+ E_\perp^*(\boldsymbol{k}',\omega')E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{J_{s-n+p}(a)}{\omega'-\overline{n-1}\Omega}\left(-\frac{\partial F_{np}}{\partial v_\perp} + \frac{n-p-1}{v_\perp}F_{np}\right) \\
&+ E_\perp^*(\boldsymbol{k}',\omega')E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{J_{s-n+p+2}(a)\, e^{-2i\alpha}}{\omega'-\overline{n-1}\Omega}\left(\frac{\partial F_{np}}{\partial v_\perp} + \frac{n-p-1}{v_\perp}F_{np}\right)\Bigg\} \\
&+ \frac{2|\boldsymbol{k}-\boldsymbol{k}'|}{\omega-\omega'}\Bigg[\{\mathrm{Re}(e^{i\bar{\alpha}}E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'))\}F_{np} \\
&\times \left(e^{i\alpha}(n-p+1)E_\perp(\boldsymbol{k}',\omega')\frac{J_{s-n+p-1}(a)}{\omega'-\overline{n+1}\Omega}\right. \\
&\left. - e^{-i\alpha}(n-p-1)E_\perp^*(\boldsymbol{k}',\omega')\frac{J_{s-n+p+1}(a)}{\omega'-\overline{n-1}\Omega}\right) \\
&+ \frac{E_z(\boldsymbol{k}',\omega')E_z(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')}{\omega'-n\Omega} \\
&\times \Bigg\{J_{s-n+p-1}(a)\exp\{i(\alpha-\bar{\alpha})\}\frac{kT}{m}\left(-\frac{\partial}{\partial v_\perp}\left(\frac{F_{np}}{v_\perp}\right) + \frac{n-p}{v_\perp^2}F_{np}\right) \\
&- J_{s-n+p+1}(a)\exp\{-i(\alpha-\bar{\alpha})\}\frac{kT}{m}\left(\frac{\partial}{\partial v_\perp}\left(\frac{F_{np}}{v_\perp}\right) + \frac{n-p}{v_\perp^2}F_{np}\right)\Bigg\}\Bigg]\Bigg)
\end{aligned} \tag{A.2}$$

where

$$F_{np} = \frac{df_0}{dv_\perp}J_n(a')J_p(a').$$

The triple sum in (A.2) can then be reduced to a double sum using the addition theorem for Bessel functions

$$\sum_{p=-\infty}^{\infty} \exp\{ip(\alpha'-\alpha)\}J_p(a')J_{\nu+p}(a) = \exp(i\nu\gamma)J_\nu(\bar{a})$$

where $\bar{a} = (|\boldsymbol{k}-\boldsymbol{k}'|v_\perp/\Omega)$ and $\gamma = 2\pi - \bar{\alpha} + \alpha = \pi + \alpha - \xi$ as shown in figure 1. Then equation (A.2) becomes

$$n^{(2)}(\boldsymbol{k},\omega)=\frac{n_0e^2\pi}{2m^2}\int v_\perp\,\mathrm{d}v_\perp\int\int\frac{\mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}'}{\epsilon(k'_\perp,\omega')\epsilon(k_\perp-k'_\perp,\omega-\omega')\epsilon(k_\perp,\omega)}$$

$$\times\sum_n\sum_s\exp\{\mathrm{i}(s\alpha-n\alpha')+\mathrm{i}(s-n)\beta\}\frac{\mathrm{J}_s(a)}{\omega-s\Omega}$$

$$\times\Bigg(\Bigg\{E_\perp(\boldsymbol{k}',\omega')E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{\mathrm{J}_{s-n-2}(\bar{a})\,\mathrm{e}^{-2\mathrm{i}\beta}}{\omega'-\overline{n+1}\Omega}\left(\frac{\partial P_n}{\partial v_\perp}-\frac{n+1}{v_\perp}P_n\right)$$

$$-E_\perp(\boldsymbol{k}',\omega')E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{\mathrm{J}_{s-n}(\bar{a})}{\omega'-\overline{n+1}\Omega}\left(\frac{\partial P_n}{\partial v_\perp}+\frac{n+1}{v_\perp}P_n\right)$$

$$+E_\perp^*(\boldsymbol{k}',\omega')E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{\mathrm{J}_{s-n}(\bar{a})}{\omega'-\overline{n-1}\Omega}\left(-\frac{\partial P_n}{\partial v_\perp}+\frac{n-1}{v_\perp}P_n\right)$$

$$+E_\perp^*(\boldsymbol{k}',\omega')E_\perp^*(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\frac{\mathrm{J}_{s-n+2}(\bar{a})\,\mathrm{e}^{2\mathrm{i}\beta}}{\omega'-\overline{n-1}\Omega}\left(\frac{\partial P_n}{\partial v_\perp}+\frac{n-1}{v_\perp}P_n\right)$$

$$+2\{\mathrm{Re}(E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\,\mathrm{e}^{\mathrm{i}\alpha'})\}\frac{a'P_n}{v_\perp}$$

$$\times\left(\mathrm{e}^{-\mathrm{i}\beta}E_\perp(\boldsymbol{k}',\omega')\frac{\mathrm{J}_{s-n-1}(\bar{a})}{\omega'-\overline{n+1}\Omega}-\mathrm{e}^{\mathrm{i}\beta}E_\perp^*(\boldsymbol{k}',\omega')\frac{\mathrm{J}_{s-n+1}(\bar{a})}{\omega'-\overline{n-1}\Omega}\right)\Bigg\}$$

$$+\frac{|\boldsymbol{k}-\boldsymbol{k}'|}{\omega-\omega'}\Bigg[2\frac{kT}{m}\frac{E_z(\boldsymbol{k}',\omega')E_z(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')}{\omega'-n\Omega}$$

$$\times\Bigg\{\exp\{-\mathrm{i}(\bar{\alpha}+\beta)\}\mathrm{J}_{s-n-1}(\bar{a})\left(-\frac{\partial}{\partial v_\perp}\left(\frac{P_n}{v_\perp}\right)+\frac{n}{v_\perp^2}P_n\right)$$

$$-\exp\{\mathrm{i}(\bar{\alpha}+\beta)\}\mathrm{J}_{s-n+1}(\bar{a})\left(\frac{\partial}{\partial v_\perp}\left(\frac{P_n}{v_\perp}\right)+\frac{n}{v_\perp^2}P_n\right)-2\mathrm{i}\sin(\alpha'-\bar{\alpha})\mathrm{J}_{s-n}(\bar{a})\frac{a'P_n}{v_\perp^2}\Bigg\}$$

$$+2\{\mathrm{Re}(E_\perp(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\,\mathrm{e}^{\mathrm{i}\bar{\alpha}})\}P_n\Bigg\{\Bigg((n+1)E_\perp(\boldsymbol{k}',\omega')\,\mathrm{e}^{-\mathrm{i}\beta}\frac{\mathrm{J}_{s-n-1}(\bar{a})}{\omega'-\overline{n+1}\Omega}$$

$$-(n-1)E_\perp^*(\boldsymbol{k}',\omega')\frac{\mathrm{e}^{\mathrm{i}\beta}\mathrm{J}_{s-n+1}(\bar{a})}{\omega'-\overline{n-1}\Omega}\Bigg)$$

$$-\frac{a'}{2}\left(\frac{E_\perp(\boldsymbol{k}',\omega')}{\omega'-\overline{n+1}\Omega}[\mathrm{e}^{\mathrm{i}\alpha'}\mathrm{J}_{s-n}(\bar{a})+\exp\{-\mathrm{i}(\alpha'+2\beta)\}\mathrm{J}_{s-n-2}(\bar{a})]\right)$$

$$+\frac{a'}{2}\left(\frac{E_\perp^*(\boldsymbol{k}',\omega')}{\omega'-\overline{n-1}\Omega}[\exp\{\mathrm{i}(\alpha'+2\beta)\}\mathrm{J}_{s-n+2}(\bar{a})+\mathrm{e}^{-\mathrm{i}\alpha'}\mathrm{J}_{s-n}(\bar{a})]\right)\Bigg\}\Bigg]\Bigg) \qquad \text{(A.3)}$$

where

$$P_n=\frac{\mathrm{d}f_0}{\mathrm{d}v_\perp}\mathrm{J}_n(a') \qquad \beta=\pi-\xi.$$

The reduction of the triple sum in (24) to the double sum in (25) makes the numerical evaluation of $n^{(2)}(\boldsymbol{k},\omega)$ easier. For the situation in which we are interested, a further

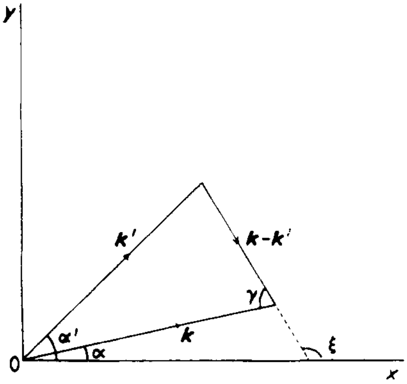

Figure 1. Geometry of the wavevectors.

simplification is possible; noting that the frequencies ω', $\omega-\omega'$ are in the optical range while Ω is typically 10^{11} Hz we may replace terms like $\omega'-\overline{n+1}\Omega$ by ω' and the dielectric functions $\epsilon(k'_\perp, \omega')$, $\epsilon(k_\perp - k'_\perp, \omega-\omega')$ by unity. It then becomes possible to contract the double summation in (A.3) to a single sum over s, with the use of the identity

$$\sum_{n=-\infty}^{\infty} \exp\{-\mathrm{i}n(\pi+\alpha'-\xi)\}\mathrm{J}_{s-n}(\bar{a})\mathrm{J}_n(a') = \mathrm{e}^{\mathrm{i}s\theta}\mathrm{J}_s(b)$$

where

$$\tan\theta = \frac{a'\sin(\alpha'-\xi)}{\bar{a}+a'\cos(\alpha'-\xi)}$$

and

$$b = (\bar{a}^2 + a'^2 - 2a'\bar{a}\cos(\alpha'-\xi))^{1/2}.$$

This then gives (22).

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